AD NUMBER	
AD012619	
CLASSIFICATION CHANGES	
TO:	**unclassified**
FROM:	**confidential**
LIMITATION CHANGES	
TO:	**Approved for public release; distribution is unlimited.**
FROM:	**Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; FEB 1953. Other requests shall be referred to Wright Air Development Center, Wright-Patterson AFB, OH 45433.**
AUTHORITY	
28 Feb 1965 per Group-4, DoDD 5200.10; Aerospace Research Lab ltr dtd 15 Apr 1968	

UNCLASSIFIED

AD NUMBER
AD012619
CLASSIFICATION CHANGES
TO: **confidential**
FROM: **secret**
AUTHORITY
28 Feb 1956 per Group-4, DoDD 5200.10

THIS PAGE IS UNCLASSIFIED

AD No. 12619
ASTIA FILE COPY

SECURITY INFORMATION

SECRET

WADC TECHNICAL NOTE WCRR 53-2 SECURITY INFORMATION

DECELERATIONS OF BALLISTIC TYPE MISSILE ON REENTRY INTO ATMOSPHERE (Secret)

Dr. M. G. Scherberg
Theodore Rubin

Aeronautical Research Laboratory

1

February 1953

RDO-460-50-25

Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

Cy #1 SECRET 53WC-18724-1

FOREWORD

Mr. J. S. Yasecke of the Bombardment Missiles Branch of the Weapons Systems Division requested the initiation of the work reviewed in this report via a suborder WCSGS-23 under Project No. R448-82 of the MX-1593 (Atlas) Project.

The authors wish to express their gratitude for the aid of Dr. J. R. Foote in the preparation of this technical note.

ABSTRACT

Calculations performed by Convair show that the velocity of the Atlas missile exceeds 19,500 ft/sec during the entire flight path with consequent severe aerodynamic heating and loading design problems. This is a report of a study performed in the Mathematics Research Branch, Aeronautical Research Laboratory on the feasibility of reducing velocities in the denser portion of the atmosphere by special flight maneuvers of the missile which increase the drag force in the less dense portions of the atmosphere. Calculations are performed to determine modifications of the Convair trajectory resulting from configuration and flight attitude changes. Results indicate that a sufficient velocity decrease may be attained provided that the angle of attack of the missile can be suitably controlled.

PUBLICATION REVIEW

This report has been reviewed and is approved.

Leslie B. Williams
LESLIE B. WILLIAMS
Colonel, USAF
Chief, Aeronautical Research Laboratory
Directorate of Research

TABLE OF CONTENTS

Cy #1

53WC-18724-1

LIST OF ILLUSTRATIONS

Introduction

Convair* has studied, and is continuing to study, the problem of developing a long range, hypersonic, offensive missile (Atlas). Their studies have prompted them to adopt a sharp, long, cone-shaped missile that is rocket propelled only during the initial stages of flight - the kinetic energy attained during this period being more than sufficient to get the missile to the target. The missile would follow a parabolic (ballistic) trajectory with the axis of the missile parallel to the velocity vector (the streamline attitude) at all times, reentering the atmosphere at an angle of 23° from the local horizontal. Under such conditions, calculations made at Convair indicate that, if this nose cone missile reenters the atmosphere at an altitude of 340,000 ft and with a velocity of 23,000 ft/sec, the velocity of the missile will be reduced to approximately 20,000 ft/sec at the time the missile reaches sea level by the aerodynamic drag forces which the missile experiences as it enters and passes through the denser portions of the atmosphere.

Because of the high speeds ($v \geq 19,500$ ft/sec) there are large structural loadings on the missile, high temperatures and large temperature gradients on the surface of the missile and a comparatively high rate of erosion of the missile surface; the maximum values of the temperature, temperature gradient and loading occuring near sea level, approximately at the altitude of detonation. Convair presently proposes a complicated transpiration cooling system to alleviate the surface difficulties, to add more structure for the loading, and to add more insulation to overcome the erosion problem. All of these solutions add non-payload weight and/or add complexities to the system.

The magnitudes of the aforementioned design problems may be taken to be indicative of the difficulty of their solution. Thus, Convair's proposed transpiration cooling system is indicative of the very high surface temperatures involved. If it were possible to keep this temperature below a certain level, which would still be considered a fairly high temperature, then the addition of more insulation might also be a solution, which is much simpler than the transpiration cooling system solution. Also, Convair's computations indicate that the missile would experience a load of 25g's during the flight. Such decelerations place very difficult design requirements on the electronic equipment, and perhaps even on the payload. An upper limit to the magnitude of the loading experienced by the missile was set at 15g's to ease these design requirements.

The magnitudes of temperature, loading, etc are proportional to the product of the air density at missile altitude and the square of the velocity of the missile. Thus these problems become more critical at lower altitudes. To ease these problems the velocity of the missile at these lower altitudes must be reduced. If one is to simplify the proposed Convair solutions and to keep within the 15g loading limit, the velocity at a lower altitude must be less than the velocity computed by Convair at the corresponding altitude. Therefore, a study was begun to determine the feasibility of getting lower velocities in the denser portions of the atmosphere.

*Convair, Project Atlas Summary Report, ZM-7-011 Jan, 1952, Vols. I and II.

CHAPTER I

METHODS OF REDUCING VELOCITY OF MISSILE IN TERMINAL PHASE OF FLIGHT

One can reduce the initial velocity of the missile so that the velocity during the final stages is lower, and thereby gain the desired end. However, the initial velocity is determined by factors other than high loadings, high temperatures, etc. and so the initial velocity cannot be altered appreciably.

A second method is to increase the drag at the higher altitudes, so that lower decelerations will be experienced in the denser atmosphere. This method depends on increasing the drag area, at the higher altitudes and thus increase the deceleration at higher altitudes and bring about lower velocities at the lower altitudes. Several modifications of the missiles flight history have been treated.

(1) Permit the missile to tumble.

(2) Let the missile and tank structure travel in a broadside attitude until a certain load limit has been reached, whereupon the tank is jettisoned and the missile continues in a streamline condition.

(3) Permit the section joining the missile to the tank structure (this section has the shape of a cone frustrum and is hereafter referred to as a skirt) to travel with the missile in a streamline condition until a certain loading limit is reached, whereupon the skirt is jettisoned and the missile continues in a streamline condition.

(4) In addition to (3), one might simultaneously decrease the entrance angle (angle between velocity vector and horizon) so that the increased drag operates over a longer distance and hence a longer period of time.

(5) Use of the fact that the drag coefficient increases with increasing angle of attack until it reaches a maximum at an angle of attack of approximately 80° (See Fig. II). The missile would change its attitude, and hence the drag force, in such a manner as to follow a prearranged loading schedule.

The criteria of success of a suggested method were that the final velocity (velocity at an altitude of approximately 4000 ft) be less than or equal to 16,000 ft/sec and that the loading at no time be greater than 15g's.

CHAPTER II

COMPUTATIONS OF DRAG ON MISSILE

Section A - <u>Environmental Conditions and Assumptions of the Computations</u>

An environmental condition that should be emphasized is the complete lack of experimental data. All the computations done are based on theoretical results and approximations. Therefore results of computations can only be used to guide the lines of investigation. One should look at numerical results as only giving order of magnitude until such time as certain assumptions are validated by experiment.

The configuration (geometry) assumed for the operational missile (nose-cone-skirt combination) is given in Fig. I.

In computations, where the attitude of the missile changes (i.e. where idea (5) is used) the following prearranged loading (angle of attack) schedule was assumed. The missile-skirt combination enters the atmosphere at an angle of attack of 90° and continues at this attitude until the load is approximately 14g's. When this load limit is reached, the angle of attack changes so as to maintain this 14g load limit. When the streamline condition is reached, the skirt is jettisoned. The skirt is detached at this point because a load limit of 15g's has been imposed on the system by design restrictions and further continuation of this missile-skirt combine in flight would result in loadings higher than this limit.

The method whereby such an attitude-time history is attained has not been considered. It is anticipated that, by controlling the position of the center of gravity, one could approximate the assumed loading schedule.

The possibility that the missile in a changing attitude trajectory might also rotate about the velocity vector (nutate) was considered. In that case the rotary motion was neglected in the derivation of the aerodynamic drag and aerodynamic moment. Since the nutational angular velocity was limited to 10 r.p.m. by design restrictions, it was felt that the effect of such nutation on these factors would be minor.

In the case of a streamline trajectory, it was assumed that the skirt would be jettisoned when a loading limit of 15g's had been reached and that then the missile would continue in its flight in a streamline position.

Two types of aerodynamics are used (1) Newtonian aerodynamics, which assumes that the air is a gas composed of discrete particles and that the air's drag is produced only by particle collision and (2) continuous aerodynamics which assumes that the air is a continuous medium and hence obeys the laws of conventional aerodynamics. The effect of angle of attack on the conventional aerodynamic drag coefficient being unknown, it was decided to use the ratio of the continuous aerodynamic drag coefficient for a streamline condition at a speed of 20,000 ft/sec* to the corresponding Newtonian drag coefficient for a streamline condition in order to convert the Newtonian drag coefficient vs angle of attack curve to the continuous drag coefficient vs angle of attack curve (see Fig. II), Thus, if

*The continuous drag coefficient at 20,000 ft/sec is smaller than the continuous drag coefficient at 23,000 ft/sec and hence the former was chosen so as to be able to bracket the results.

$$R = \frac{\{C_D(\text{Cont})\}\text{Streamline}}{\{C_D(\text{Newt})\}\text{Streamline}}, \quad \text{then} \{C_D(\alpha)\}_{\text{Cont}} = R\{C_D(\alpha)\}\text{Newt}.$$

where C_D is the symbol for the drag coefficient. Thus, in Fig. II the lower curve is obtained merely by multiplying the higher curve by this constant factor R.

Calculations indicate that $R \cong 1/3$. Hence, one may reduce the total number of computations necessary, by noting that if a method is successful using the continuous drag coefficient, the method will also be successful using the Newtonian drag coefficient, since the latter means a larger drag, hence a greater deceleration, and hence a smaller final velocity. Similarly, if a method fails using the Newtonian drag coefficient, the method will also fail using the continuous drag coefficient.

Section B. - Method of Drag Computation

The expression* derived for the Newtonian drag coefficient of a simple cone, using the maximum circular cross sectional area as a basis is

$$(1) \quad C_D(\alpha) = \frac{h\cos^2\phi}{r\pi}\int_{-\theta_L}^{\pi+\theta_L}(\sin\alpha \cdot \sin\theta + \cos\alpha\tan\phi)^3 d\theta$$

where h = height of cone, ft.
r = radius of base, ft.
α = angle of attack.
ϕ = half angle of cone
$\theta_L = \sin^{-1}(\operatorname{ctn}\alpha\tan\phi)$, radians

If one sets a = $\sin\alpha$ and b = $\cos\alpha\tan\phi$, then

$$(2) \quad C_D(\alpha) = \frac{h\cos^2\phi}{\pi r}\int_{-\theta_L}^{\pi+\theta_L}(a^3\sin^3\theta + 3a^2b\sin^2\theta + 3ab^2\sin\theta + b^3)\,d\theta$$

Using the formulae

$$\int\sin^3\theta d\theta = -\frac{1}{3}\cos\theta\left[\sin^2\theta + 2\right]; \quad \int\sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}; \quad \int\sin\theta\, d\theta = -\cos\theta$$

the drag coefficient may then be evaluated as a function of the angle of attack.

The missile-skirt combination, diagrammed in Fig. I, is not just a simple cone. Therefore a modification is introduced in order to obtain the Newtonian drag coefficient of this operational missile.

*This expression is derived in Appendix A.

ASSUMED MISSILE GEOMETRY

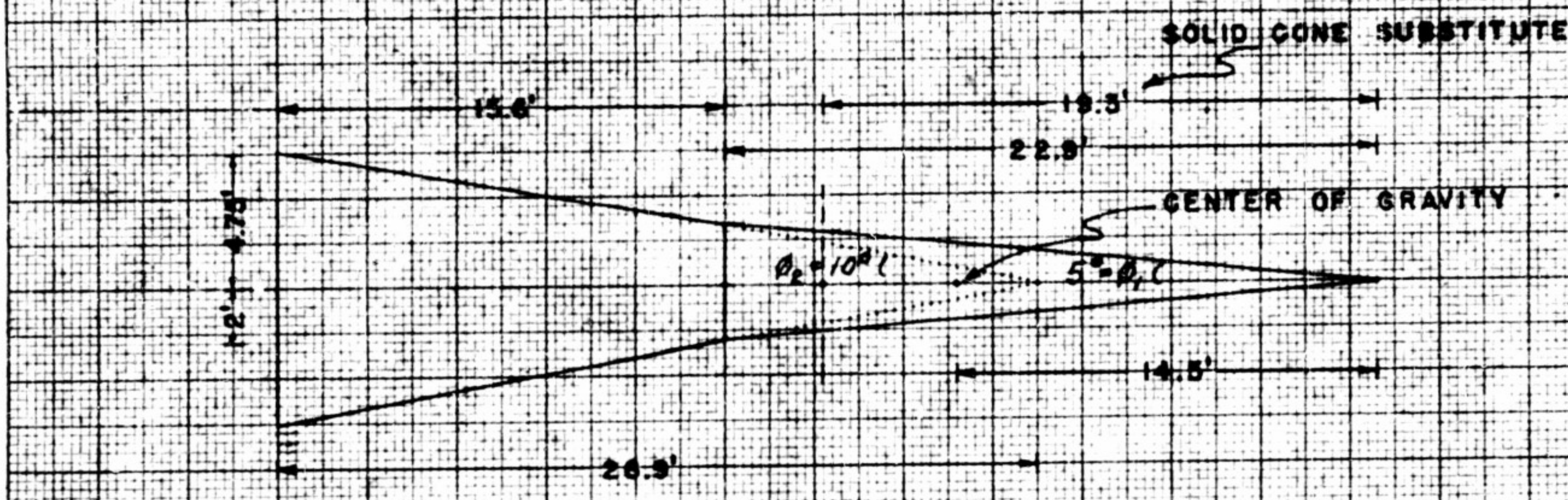


FIGURE I

SECRET

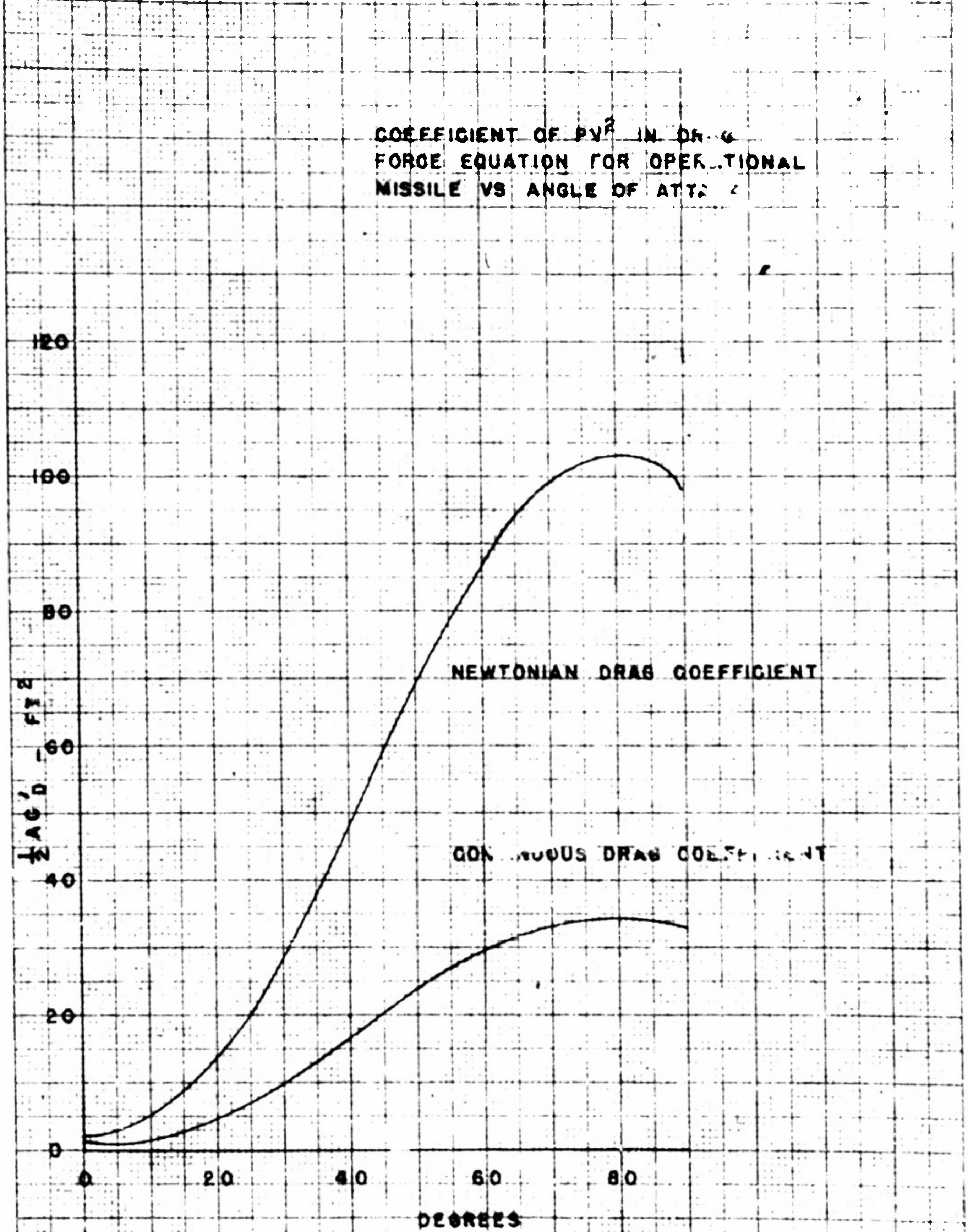


FIGURE II

SECRET

Let cone I be the missile nose cone; cone II the cone from which the skirt is derived; cone III the cone that remains after the skirt has been removed from cone II. Cone I, II, and III being simple cones, their drag coefficients can be computed by placing their respective physical constants (dimensions) in the equation for $C_D(\alpha)$. From these coefficients, the drag force on each cone may be determined. The drag coefficient of the operational missile, diagrammed in Fig. II, is then obtained from the equation

$$\frac{1}{2} A' \rho v^2 C_D'(\alpha) = F_I + F_{II} + F_{III}$$

where A' = area of base of skirt - ft^2; F_I = drag force on cone I;
ρ = air density - $slugs/ft^3$; F_{II} = drag force on cone II;
v = velocity of missile - ft/sec; $-F_{III}$ = drag force on cone III;
C_D' = drag coefficient of operational missile (see Fig. II)

Having the drag coefficient as a function of α, the drag force can then be determined. The lifting force does not have an important influence on the trajectory as far as velocity effects are concerned.

The differential equations to be integrated were $m\ddot{x} = F_x$ and $m\ddot{z} = F_z - mg$ where F_x and F_z are components of the drag force, $\ddot{x}$ and $\ddot{z}$ are components of the acceleration, m is the mass of the operational missile; and g is the gravitational constant. The details of the computation may be found in Appendix B.

CHAPTER III

COMPUTATIONAL RESULTS AND DISCUSSION OF RESULTS

Computations were started on the first idea - that of permitting the missile to tumble. However, errors made in the derivation rendered the results worthless. The results of such a tumbling calculation would yield a final velocity smaller than that yielded by the streamline trajectory - for the streamline trajectory has minimum wetted area and hence minimum drag over the entire flight path, while the tumbling by increasing the "wetted" area would permit a greater drag to be experienced at least part of the time. Also, the results of such a tumbling computation would yield a final velocity larger than that yielded by using the change-of-attitude-to-produce-a-prearranged-loading-schedule idea (idea (5)), for the latter has maximum drag consistent with the definition of a successful run, while the former will have a smaller "wetted" area present at least part of the time and hence a smaller average drag. Thus, the results of tumbling are expected to be bracketed above by the streamline results and below by the results of the changing attitude trajectory. Therefore, computations on the tumbling idea were not renewed.

Computations on the second idea - that of permitting the missile and tank structure to travel in a broadside condition - were performed, and the results indicated that the idea failed for the final velocity was too high ($v \geq$ 19,000 ft/sec).

The Computation Branch of Flight Research Laboratory performed I.B.M. computations on the following trajectories:

(1) A streamline trajectory of the operational missile using Newtonian aerodynamics with an entrance angle of 23°.

(2) The same as (1) except that an entrance angle of 14° was used.

(3) A changing attitude trajectory of the operational missile assuming the loading history previously discussed.

(4) The same as (3) except that continuous aerodynamics was used.

The numerical results of the computations are summarized in Table 1. (p. 8).

These results indicate that the streamline trajectories do not yield a sufficient velocity decrease. Using a previous argument it can be seen that a computation of a streamline trajectory using continuous aerodynamics would also produce an unsuccessful run. The changing attitude trajectories, however, do indicate sufficient velocity decreases, even though the success of the run using continuous aerodynamics (Case 4 of Table 1) is marginal. Here one could increase the margin of success by using a shallower entrance angle so that the drag operated over a longer period of time thus producing an even larger velocity decrease.

One can also note that in the case of the two changing attitude trajectories, the final velocities differ by a comparatively small amount (15,000 ft/sec as compared to 14,000 ft/sec) while the drag coefficients used differ by a factor of three. This would indicate that the final result is not strongly dependent on the drag coefficient.

In Table II there is a tabulation of the angle of attack vs time history for a changing attitude trajectory using Newtonian aerodynamics and an entrance angle of 23°. If one plots the data, the shape of the curve is characteristic of the assumed loading schedule.

Time (sec)	0	10	19	20	21	22	23	24	25
Angle of Attack	80°	80°	80°	62.5°	51°	41.5°	33.3°	27.5°	22.7°

Time (sec)	26	27	28	29	30	31	32	33
Angle of Attack	19.0°	15.2°	12.3°	10.0°	7.7°	5°	3°	0°

TABLE II

Since the loadings and temperatures experienced by the operational missile are proportional to V^2 (by assumption), these problems have been reduced by a factor of about 50%, by the reduction of the velocity from the original 23,000 to the final velocity of 15,500 ft/sec. How the problem of surface erosion would be affected is unknown, but presumably the rate of surface erosion would be smaller.

Time	Altitude	Velocity	Load	Computed Density	Remarks
	Case (1) - Streamline Trajectory - Newtonian drag coefficient - Entrance Angle of 23°				
0 sec	340,000 ft	23,000 ft/sec	1.00g's	4.167×10^{-9} slugs/ft^3	
28	76,572	22,565	13.3	1.015×10^{-4}	Skirt rejected
29	67,098	22,138	1.22	1.597×10^{-4}	
36	1,656	20,987	15.53	2.274×10^{-3}	
	Case (2) - Streamline Trajectory - Newtonian drag coefficient - Entrance Angle of 14°				
0 sec	340,000 ft	23,000 ft/sec	1.00g's	4.167×10^{-9} slugs/ft^3	
43	72,486	21,891	15.40	1.234×10^{-4}	Skirt rejected
44	66,030	21,397	1.33	1.680×10^{-4}	
54	2,090	19,761	13.68	2.248×10^{-3}	
	Case (3) - Angle of attack changes - Newtonian drag coefficient is used Entrance angle of 23°				
0 sec	340,000 ft	23,000 ft/sec	.98g's	4.167×10^{-9} slugs/ft^3	
19	165,513	21,762	13.14	1.941×10^{-6}	Change of angle of attack begins*
34	55,609	15,388	14.24	2.764×10^{-4}	Skirt rejected
35	49,024	14,931	1.07	2.[illegible]79$\times 10^{-4}$	
41	3,540	14,295	6.[illegible]3	3.159×10^{-3}	
	Case (4) - Angle of attack changes - conventional drag coefficient used. Entrance angle 23°				
0 sec	340,000 ft	23,000 ft/sec	.996g's	4.167×10^{-9} slugs/ft^3	
21	136,266	22,052	1.40	5.86×10^{-6}	Change of angle of attack begins
34	37,964	16,566	14.24	5.828×10^{-4}	Skirt rejected
35	30,872	16,108	3.11	8.261×10^{-4}	
39	3,087	15,498	7.97	2.137×10^{-3}	

Summary of Results of Trajectory Computations

TABLE I

*Prior to and including the time indicated the angle of attack has been held constant. Afterwards the angle of attack begins to change so as to keep the prearranged loading schedule already discussed.

CHAPTER IV

DISCUSSION OF CONTROL OF ANGLE OF ATTACK

Section A - The Aerodynamic Moment

Successful runs theoretically can be achieved provided that there is the ability to control the angle of attack. In a sense, therefore, the problems mentioned originally have been replaced by the single problem of control. Note that this does not mean that loading and temperature problems are completely removed, but that they are reduced.

Since the problem of controlling the angle of attack can only be solved if much more design data, as e.g. the weight distribution, is available, only general or order of magnitude statements about this problem will be made in this report. Because of this fact many assumptions and simplifications will be made.

The aerodynamic moment, which tends to streamline the attitude of the operational missile, is a part of this control problem. A subsidiary computation of this moment for a cone at an angle of attack of 90° was made. This broadside condition was used because (1) the computation was simplified and (2) the computed moment for this condition would be close to the maximum value of the aerodynamic moment, as was true for the drag coefficient, and (3) only order of magnitude results were of interest.

Newtonian aerodynamics is used to compute this aerodynamic moment. To compute the moment, M, for a simple cone, one has to compute $\int[\vec{r}\times\vec{F}_{da}]\,da$ where the integration is performed over the "wetted" surface area of the cone, $\vec{r}$ is a vector from the center of rotation to the surface area element da, and $\vec{F}_{da}$ is the force on a surface area element.

Now $$\vec{M} = M_{x'}\vec{i'} + M_{y'}\vec{j'} + M_{z'}\vec{k'}$$

From the symmetry properties of the cone
$$M_{y'} = M_{z'} = 0$$

Therefore, $$M = M_{x'}\vec{i'}\ ;\quad M_{x'} = \int[\vec{r}\times\vec{F}_{da}]_{x'}\,da = \int[y'(F_{da})_{z'} - z'(F_{da})_{y'}]\,da$$

where y', z', $(F_{da})_{y'}$ and $(F_{da})_{z'}$ are components of $\vec{r}$ and $\vec{F}_{da}$ respectively. Using the geometry, and noting that $z' = z - z_0$ and that $R = (h-z)\tan\phi$ one obtains the following:*

$$M_{x'} = \frac{4}{3}\rho v^2\left[-\sin^2\phi\tan\phi\int_0^{z_0}(h-z)^2dz + \sin\phi\cos\phi\int_0^{z_0}(z-z_0)(h-z)dz\right]$$
$$+\frac{4}{3}\rho v^2\left[-\sin^2\phi\tan\phi\int_{z_0}^{h}(h-z)^2dz + \sin\phi\cos\phi\int_0^{z_0}(z-z_0)(h-z)dz\right]$$

*The derivation of this expression can be found in Appendix C.

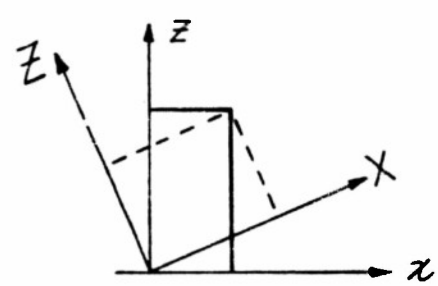
where the convention that a torque producing a counterclockwise rotation has a positive sign has been adopted and where z_o is the distance from the base of the cone to the assumed center of rotation.

Again modifications are made because the operational missile is not just a simple cone. The results obtained for various centers of rotation are listed below:

Center of Rotation	Distance from base of Skirt	Aerodynamic Moment
Center of Gravity	24'	$-1245 \rho V^2$
Junction of Skirt and Cone	15.6'	$-455 \rho V^2$
Center of Pressure	11.8'	0

Section 5 - Discussion of Restoring Moments*

It has been stated previously that the possibility that the missile would have a nutational velocity would be considered. Such a velocity has an associated centripetal force. One component of this force produces a centripetal restoring moment, whose magnitude is $\omega^2 I_{xz}$ where ω is the rate of nutation in radians/sec and I_{xz} is the product of inertia. In computing I_{xz}, the operational missile was replaced by a uniform solid cone having the same weight, same half angle as the missile, and having a center of gravity whose distance from the tip of the cone is the same as the distance between the center of gravity and apex of the operational missile. In Fig. I the length of this substitute cone is indicated.



$$X = x \cos\alpha + z \sin\alpha$$

$$Z = -x \sin\alpha + z \cos\alpha$$

Fig. III

Axes For Determining I_{xz}

*The aerodynamic moment for $0 < \alpha < \frac{\pi}{2}$ tends to decrease α (angle of attack). A restoring moment is one tending to increase α.

The product of inertia is then $\int XZ\,dm = \rho \int XZ\,dv$

where ρ is the density of the uniform cone and the integration is performed over the volume of this uniform cone. The coordinates (X,Z) and their relationship to the body coordinates of a mass point of the cone are defined in the diagram. The center of the body coordinate system is taken to be the center of rotation. The y coordinates in the two systems i.e., y and Y, are not indicated for they are the same, i.e. y = Y.

For ω = 10 r.p.m., the limiting nutational angular velocity, the restoring moment has a magnitude of 1552 sin 2α (slug ft^2), where the center of rotation has been taken to be the center of gravity and where α is the angle of attack.

Another force tending to produce a torque which increases the angle of attack is the inertial force i.e. the reaction to the drag force. The magnitude of this force is Mal; M is the mass of the cone in slugs; a is the deceleration in ft/sec^2; and l is the perpendicular distance, in feet, between the center of rotation and the center of gravity. The center of rotation shall be taken to be the center of gravity throughout the flight. Hence, this torque has zero magnitude.

It might be that the center of rotation does not coincide with the center of gravity throughout the flight, but this possibility shall be ignored in view of the policy of finding the order of magnitude of desired results.

As an example of these order of magnitude results that can be attained, consider the following: Let the missile have an angle of attack of 80° and a nutating angular velocity of 10 r.p.m. The center of rotation is taken to be the center of gravity. Assuming that the aerodynamic moment for the 80° and 90° angles of attack are nearly equal, (in the name of order of magnitude computations), the aerodynamic moment is $1245\rho_2 v^2$ for this 80° angle of attack. The restoring moment, is then 531 slug/ft^2. Assuming that $v^2 \sim 5 \times 10^8$, for the aerodynamic moment to balance the restoring moment, $1245 \times 5 \times 10^8 \rho = 531$ or $\rho = 8.5 \times 10^{-10}$ slugs/ft^3. This density corresponds to an altitude of approximately 390,000 ft.

Even though this calculation is extremely rough, it indicates that, for the major part of the terminal portion of the flight, the restoring moment would be less than the aerodynamic moment and hence may be used to help control the angle of attack.

CHAPTER V

CONCLUSIONS

The computations discussed in this report are based on theoretical results and approximations, so that results can only be used to guide the lines of investigation. The report is to be considered as the initial phase of the study of the feasibility of reducing the velocity of the missile still further by having the missile experience higher drag in less dense portions of the atmosphere. The results of the computations discussed in this report seem to indicate that the desired result can be attained and hence further investigation of the problem should proceed.

APPENDIX A

DERIVATION OF DRAG FORCE EQUAT..

The missile is assumed to be standing still wi... ...ne air molecules all having a velocity equal and opposite to the veloci... ...ector of the missile. It is assumed that (1) the element of force acting o... the missile is the time rate of change of momentum along the normal to ...e surface area element and (2) each molecule colliding with the missile has its direction altered so that its new velocity vector lies in the plane tangent to the surface at the point of collision. In consequence of these two assumptions only the normal compon... of the initial velocity vector need be considered in the derivation.

If φ is the angle between the plane perpendicular to the velocity vector and the normal to the surface, then $v \sin\varphi$ is the component of velocity along this normal. Furthermore, the projection of this normal component onto the original velocity vector is the only component contributing to the drag force. This projection has magnitude $v \sin^2\varphi$. If M_0 is the mass delivered per unit time on a surface area element, da, then $M_0 v \sin^2\varphi$ is the drag force, F_{da}, on such an area element. The angle φ is now to be determined as a function of angle of attack and position of the area element.

Consider Figure A-1. In it the lines (6,10), (6,11), (5,12) and (3,4) are parallel. Also the lines (5,6) and (11,12) are parallel. The calculation of φ as a function of α' and θ can now proceed, where $\alpha + \alpha' = \pi/2$, α ... being the angle of attack.

Let the distance between two points, p and q, be denoted by d(p,q). In particular let d(1,8)=h, h being the altitude ... ...e cone. Now

$$\sin\varphi = \frac{d(5,6)}{d(2,6)}$$

First d(5,6) is determined:

$$d(5,6) = d(11,12) = d(4,11)\cos\alpha' = \cos\alpha'\left[d(\ldots 4) - d(8,11)\right]$$

$$= \cos\alpha'\left[h\tan\alpha' - d(6,8)\sin\theta\right] = \cos\ldots\left[\ldots\tan\alpha' - h\tan\phi\sin\theta\right]$$

Next d(2,6) is determined:

$$d(2,6) = h\sec\phi\tan(\nu-\phi) = h\sec\phi\,\frac{\tan\nu - \ldots\phi}{1+\tan\nu\ldots\tan\phi}$$

But, $$\tan\nu = \frac{d(3,8)}{h} = \frac{d(4,8)\csc\theta}{h} = \tan\alpha'\ldots\theta$$

Or

$$d(2,6) = h\sec\phi\,\frac{\tan\alpha'\csc\theta - \tan\phi}{1+\tan\alpha'\csc\theta\tan\phi} = \ldots\sec\phi\,\frac{\tan\alpha' - \sin\theta\tan\phi}{\sin\theta + \tan\alpha'\tan\phi}$$

therefore

$$\sin\varphi = \frac{d(5,6)}{d(2,6)} = \frac{\cos\alpha'}{\sec\phi}(\sin\theta + \tan\alpha'\tan\phi)$$

or finally

$$\sin\varphi = \cos\phi\,(\cos\alpha'\sin\theta + \tan\phi\,\sin\alpha')$$

To determine the mass transfer on the triangular area element whose center line is (1,6), one multiplies the component of velocity along the normal to the surface, the area of the triangular element, and the air density at the altitude of the missile i.e.

$$M_o = \rho v \sin\varphi\, dA = \rho v \sin\varphi\, \tfrac{1}{2} h \sec\phi\, r\, d\theta$$

The element of force, F_{da}, is then

$$F_{da} = \tfrac{1}{2}\rho v^2 h r \sec\phi\, \sin^3\varphi\, d\theta$$

Substituting the derived expression for $\sin\varphi$ into F_{da}, one has for the element of drag force:

$$F_{da} = \frac{1}{2}\rho v^2 h r \sec\phi \cos^3\phi\,(\cos\alpha'\sin\theta + \sin\alpha'\tan\phi)^3$$

Hence the total force is

$$F = \frac{\rho v^2}{2} h r \cos^2\phi \int_{-\theta_L}^{\pi+\theta_L} (\cos\alpha'\sin\theta + \sin\alpha'\tan\phi)^3\, d\theta$$

where θ_L is a limiting angle and is given by the equation $\sin\theta_L = \tan\alpha'\tan\phi$.

To prove this equation, consider Figure A-2. The point P bisects the line AB, which is drawn from A parallel to the velocity vector $\vec{V}$. A line is drawn from the apex of the cone to P and then extended until it intersects the base. Then t is the distance from this point of intersection to the center of the base; h is the height of the cone; r is the radius of the base; ϕ is the half angle; α' is the angle that the plane perpendicular to the velocity vector makes with the axis of symmetry of the cone; and σ is the angle between the symmetry axis of the cone and the line joining the apex and P.

The proof then follows:

(1) $$\frac{m}{\sin\alpha'} = \frac{2r}{\cos(\alpha'-\phi)} \Rightarrow m = \frac{2r\sin\alpha'}{\cos(\alpha'-\phi)}$$

(2) $$q = h\sec\phi - m = r\,\mathrm{ctn}\,\phi\,\sec\phi - m = r\,\frac{\cos(\alpha'+\phi)}{\sin\phi\,\cos(\alpha'-\phi)}$$

(3) $$\frac{k}{q} = \frac{\sin\phi}{\cos\alpha'} \Rightarrow k = r\,\frac{\cos(\alpha'+\phi)}{\cos\alpha'\cos(\alpha'-\phi)}.$$

(4) $$\frac{x}{2r} = \frac{\cos\phi}{\cos(\alpha'-\phi)} \Rightarrow x = \frac{2r\cos\phi}{\cos(\alpha'-\phi)}$$

(5) $$u = \frac{x}{2} - k = \frac{r\sin\alpha'\sin\phi}{\cos\alpha'\cos(\alpha'-\phi)}$$

(6) $l = r \tan\alpha'$

(7) $h - l = r(\operatorname{ctn}\phi - \tan\alpha')$

(8) $\dfrac{\cos(\alpha'+\sigma)}{h-l} = \dfrac{\sin\sigma}{u} \Rightarrow \cos\alpha' \operatorname{ctn}\sigma - \sin\alpha' = \dfrac{\cos^2\phi\cos^2\alpha' - \sin^2\phi\sin^2\alpha'}{\sin\alpha'\sin^2\phi}$

(9) $\operatorname{ctn}\sigma = \operatorname{ctn}^2\phi \operatorname{ctn}\alpha'$

(10) $\dfrac{t}{h} = \tan\sigma = \tan^2\phi \tan\alpha'$

(11) $\sin\theta_L = \dfrac{t}{r} = \dfrac{t}{h\tan\phi} = \tan\phi\tan\alpha'$

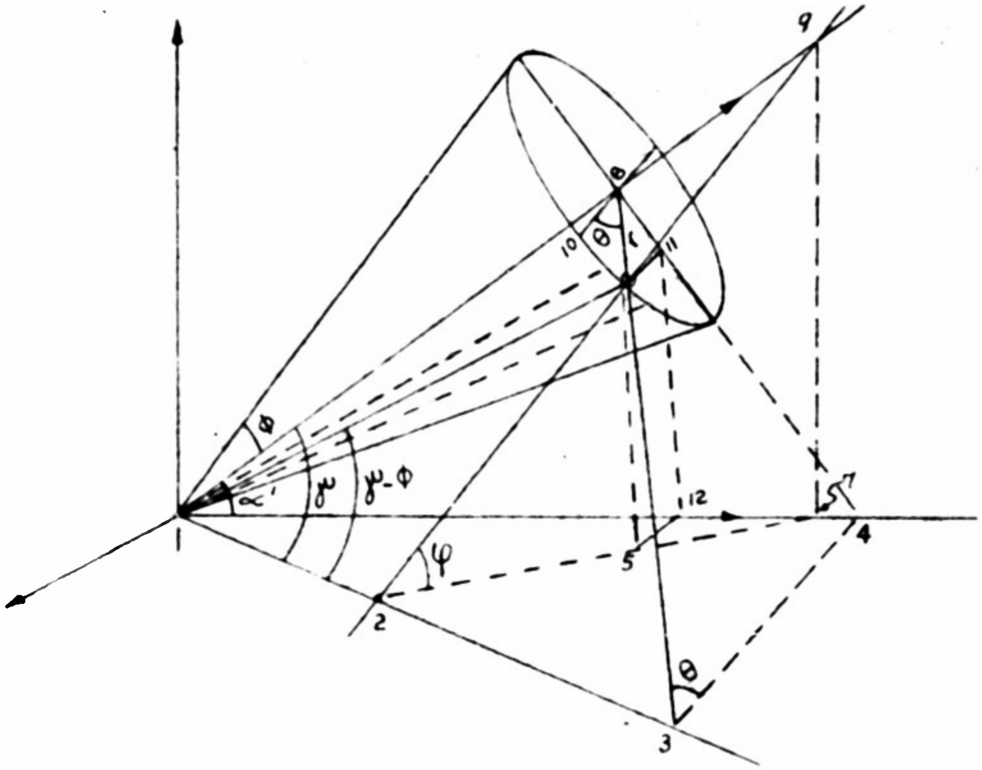
This completes the proof.

If α is the angle of attack, noting that $\alpha + \alpha' = \frac{\pi}{2}$:

$$\frac{\rho v^2}{2} C_D(\alpha) = \frac{\rho v^2}{2} \frac{h\cos^2\phi}{\pi r} \int_{-\theta_L}^{\pi+\theta_L} (\sin\alpha\sin\theta + \cos\alpha\tan\phi)^3 d\theta$$

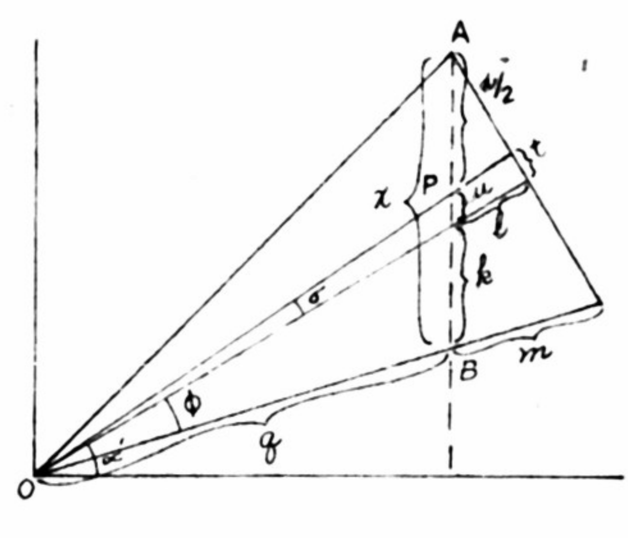
where

$$\theta_L = \sin^{-1}(\operatorname{ctn}\alpha \tan\phi)$$



GEOMETRY FOR DETERMINING φ.

Figure A-1



GEOMETRY FOR DETERMINING $\sin \theta_L = \dfrac{t}{\text{Radius of Base of Cone}}$.

Figure A-2

APPENDIX B

DETAILS OF TRAJECTORY COMPUTATIONS

Now, since the drag force for any angle of attack can be determined, one can determine the deceleration at any time. Integrating the deceleration yields the velocity and a second integration yields the position. The vector differential equation is numerically integrated in component form. These component equations are

$$m\ddot{x} = F_x \qquad m\ddot{z} = F_z - mg$$

where the $(\ddot{x},\ddot{z})$ are components of acceleration; (F_x, F_z) are components of the drag force, each being given the proper sign, m is the mass of the cone; and g is the gravitational constant 32.2 ft/sec^2. Since the only coordinate of interest is the altitude of the missile, only the equation in z is integrated twice.

A time difference, t_h, is assumed for the numerical integration - for the computations discussed in this report t_h was taken to be 1 second. The initial conditions taken are v_o = 23,000 ft/sec; η_o = (23°;14°); z_o = 340,000 ft; x_o = 0; and $\alpha = \alpha_o$. This implies that the initial velocity components $\dot{x}_o$ and $\dot{z}_o$ are $\dot{x}_o = v_o \cos \eta_o$ and $\dot{z}_o = v_o \sin \eta_o$. The computation now proceeds:

(1) Compute $V_o^2 = \dot{x}_o^2 + \dot{z}_o^2$

(2) Compute $F_o = \frac{1}{2}\rho\left(z_o + \dot{z}_o\frac{t_h}{2}\right) V_o^2 A' C_D'(\alpha_o)$

(3) Compute $\eta_o = \tan^{-1}\frac{\dot{z}_o}{\dot{x}_o}$

(4) Compute $(F_o)_x = F_o \cos \eta_o$; $(F_o)_z = F_o \sin \eta_o$

(5) Compute $\ddot{x}_o = \frac{(F_o)_x}{m}$; $\ddot{z}_o = \frac{1}{m}\left[(F_o)_z - mg\right]$

(6) Compute loading in g's $\equiv l_o = \frac{1}{32.2}\left[\ddot{x}_o^2 + \ddot{z}_o^2\right]^{1/2}$

(7) Compute $\dot{z}_1 = \dot{z}_o + \ddot{z}_o t_h$

(8) Compute $\dot{x}_1 = \dot{x}_o + \ddot{x}_o t_h$

(9) Compute $z_1 = z_o + \dot{z}_o t_h + \ddot{z}_o \frac{t_h^2}{2}$

(10) Compute $V_1^2 = \dot{x}_1^2 + \dot{z}_1^2$

(11) Compute $F_1 = \frac{1}{2}\rho\left(z_1 + \dot{z}_1\frac{t_h}{2}\right) V_1^2 A' C_D(\alpha_1)$

The stages (3) through (9) are then repeated for the second (and later) time intervals. The problem is then to determine $C_D(\alpha_n)$ where n is a running index. (The computations end when an altitude of about 4,000 feet has been reached.) In the case of a streamline trajectory $C_D'(\alpha_n) = C_D'(\alpha_o) = C_D'(0)$

for all n. In the case of a changing attitude trajectory the maximum value of $C_D'(\alpha)$ was used until a load limit of 14g's had been reached. At this point the computations were modified. The drag force, F, was then assumed to have a constant value, F_c, so that the load was close to but less than the 15g load limit. Knowing z_n, etc., then

$$C_D'(\alpha_n) = \frac{F_c}{\frac{1}{2}\rho(z_n + \dot{z}_n \frac{t_h}{2}) V_n^2 A'}$$

was computed. The corresponding α_n was then determined from the graph of C_D' vs α. This could be done easily since, for $0 \le \alpha \le \sim 83°$, C_D' is a monotonically increasing function of α. The values of α listed in Table II for $t > 19$ were determined in such a manner.

Suppose that at $t = nt_h$, the loading is about 14g's i.e.

$$\frac{1}{32.2}\left[\ddot{X}_n^2 + \ddot{Z}_n^2\right]^{1/2} \simeq 14$$

Compute $\dot{Z}_{n+1}, \dot{X}_{n+1}, Z_{n+1}$ Then

(1) $$C_D'(\alpha_{n+1}) = \frac{F_c}{\frac{1}{2}\rho(z_{n+1} + \dot{z}_{n+1}\frac{t_h}{2}) V_{n+1}^2 A'}$$

(2) $$(F_x)_{n+1} = F_c \cos \eta_{n+1} \quad \text{where } \eta_{n+1} = \tan^{-1}\frac{\dot{z}_{n+1}}{\dot{x}_{n+1}}$$

(3) $$(F_z)_{n+1} = F_c \sin \eta_{n+1}$$

(4) $$\ddot{X}_{n+1} = \frac{(F_x)_{n+1}}{m}$$

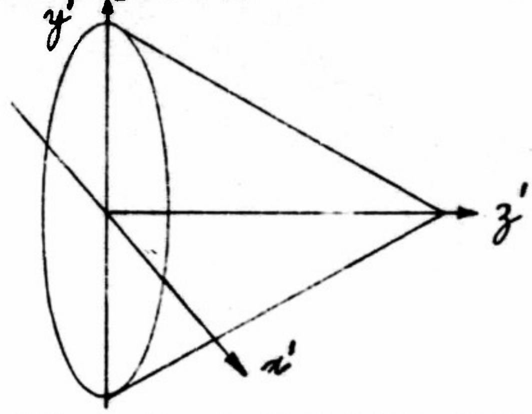
(5) $$\ddot{Z}_{n+1} = \frac{1}{m}\left[(F_z)_{n+1} - mg\right]$$

This sample of the calculations illustrates the procedure which was used.

APPENDIX C

DERIVATION OF MOMENT ABOUT A LINE PERPENDICULAR TO THE AXIS OF SYMMETRY, ANGLE OF ATTACK OF 90°

Let the coordinate system fixed in the cone be (x', y', z') axes, respectively, as shown in Figure C-1.



Coordinate Axes in Moment Calculation

Figure C-1

Let ($\bar{i}'$, $\bar{j}'$, $\bar{k}'$) be unit vectors in the (x', y', z') directions respectively.

A vector $\vec{p}$ can then be written as

$$\vec{p} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$$

Let $\vec{n}$ be a unit vector, normal to the surface making an angle φ with the (x',z') plane and let the projection of this vector onto the (x',z') plane make an angle σ with the x axis. Note that the (x',z') plane is in this case the plane perpendicular to $\vec{v}$, the velocity vector. Now

$$\vec{n} = \cos\varphi \cos\sigma\, \vec{i}' + \sin\varphi\, \vec{j}' + \cos\varphi \sin\sigma\, \vec{k}'$$

Let a point on the surface of the cone be given by the coordinates (x',y',z'). The cone is now sliced into circular elements by planes parallel to the base of the cone. The radius of the circular element on which the point (x',y',z') appears is given by

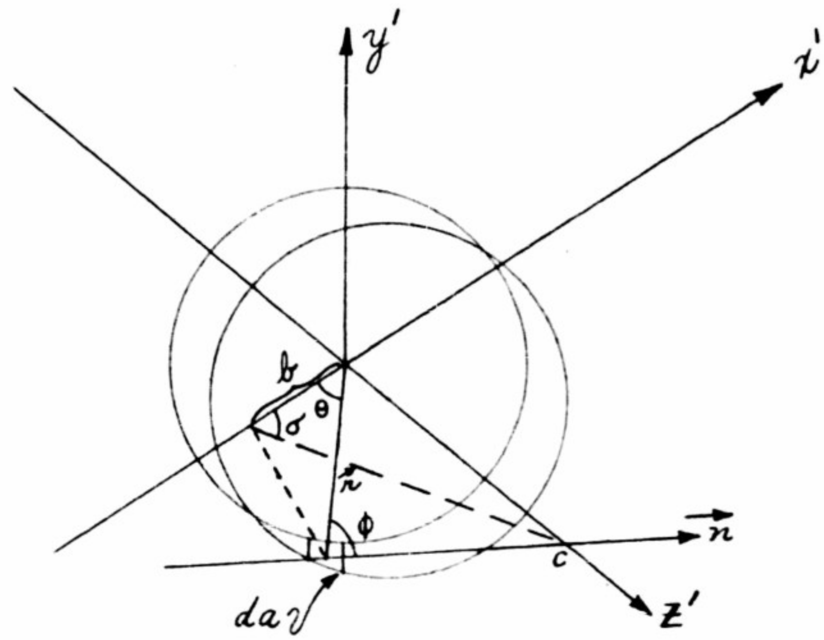
$$r = \sqrt{x'^2 + y'^2} = (h - z')\tan\phi$$

where ϕ is the half angle of the cone.

Let the point (x',y',z') be the center of an area element da which is obtained by dividing the circular elements further by angular subdivisions.

If θ is the angle which the radial vector, in the plane z' = constant, to da, the area element, makes with the x' axis, then $\tan\theta = \frac{y'}{x'}$.

Let σ be the angle between the x' axis and the projection of the normal to da onto the (x',z') plane. The angle σ is now to be determined. In Fig. C-2, r and **b** lie in the plane of the circular cross section and hence are perpendicular to the z' axis. Also note that **b** = r cos θ.



GEOMETRY FOR DETERMINING σ

Figure C-2

The angle between $\vec{r}$ and $\vec{n}$, by properties of a cone, is ϕ. In right triangle DBC, c = r tan ϕ. In right triangle ABC, it is seen that

$$\tan\sigma = \frac{c}{b} = \sec\theta \tan\phi$$

The length of the line AC is given by the expression:

$$|AC| = \sqrt{c^2 + b^2} = r\sqrt{\cos^2\theta + \tan^2\phi}$$

Hence

$$\sin\sigma = \frac{\tan\phi}{\sqrt{\cos^2\theta + \tan^2\phi}} \quad ; \quad \cos\sigma = \frac{\cos\theta}{\sqrt{\cos^2\theta + \tan^2\phi}}$$

By Appendix A,

$$\sin\varphi = \sin\theta\cos\phi \quad \text{for } \alpha = \frac{\pi}{2}$$

Then the unit vector normal to the surface, n, may be written:

$$\vec{n} = \frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{\sqrt{\cos^2\theta + \tan^2\phi}}\cos\theta\,\vec{i'} + \sin\theta\cos\varphi\,\vec{j'} + \frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{\sqrt{\cos^2\theta + \tan^2\phi}}\tan\phi\,\vec{k'}$$

This expression may be simplified:

$$1 - \sin^2\theta \cos^2\phi = 1 - \cos^2\phi + \cos^2\theta \cos^2\phi = \sin^2\phi + \cos^2\theta \cos^2\phi$$
$$= \cos^2\phi \left(\tan^2\phi + \cos^2\theta \right)$$

the expression for $\bar{n}$ then becomes:

$$\vec{n} = \cos\phi \cos\theta\, \vec{i'} + \cos\phi \sin\theta\, \vec{j'} + \sin\phi\, \vec{k'}$$

Now the moment about the x' axis can be computed:

$$(M_{x'})_{da} = z'(F_{y'})_{da} - y'(F_{z'})_{da}$$

The element of force F_{da}, acting on the area element da, is the change in momentum per unit time along the normal to da. When a molecule strikes da, it is assumed that (1) the component of the molecule's momentum in the direction of n, the inward drawn normal to the surface of the cone at the point of collision, is transferred to the body, and (2) the direction of transfer is along n. The magnitude of the momentum transfer for one molecular collision is m' v sin φ and in direction $\vec{n}$, where m' is the mass of a molecule.

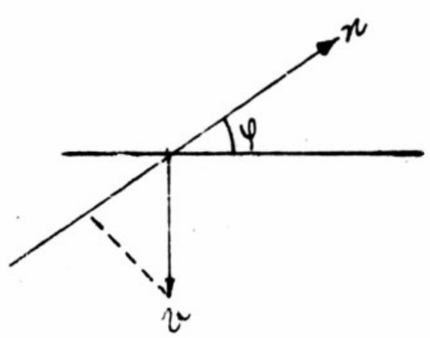
Therefore $F_{da} = v \sin\varphi M_o \vec{n}$, where M_o again is the molecular mass delivery per unit time on da. Then $M_o = \rho v (da)'$ where (da)' is the projection of da onto the (x',z') plane - a plane normal to the velocity vector of the cone. Now $(da)' = r \sin\theta\, d\theta\, dz'$ (see Fig. C-4 and C-5).

Thus

$$\vec{F}_{da} = \rho v^2 \cos\phi\, r \sin^2\theta\, d\theta\, dz'\, \vec{n}$$
$$= \rho v^2 \sin\phi \sin^2\theta\, d\theta\, (h - z')\, dz'$$

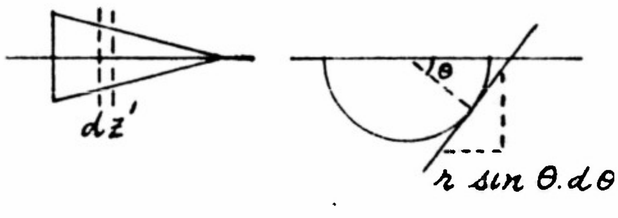
The components of F_{da} are:

$$(F_{da})_{y'} = \rho v^2 \sin\phi \cos\phi \sin^3\theta\, d\theta\, (h - z')\, dz'$$
$$(F_{da})_{z'} = \rho v^2 \sin^2\phi \sin^2\theta\, d\theta\, (h - z')\, dz'$$



COMPONENT OF VELOCITY TRANSFERRED ALONG NORMAL

Fig. C-3



DETERMINING AREA OF AREA ELEMENT

Fig. C-4

Now

$$z'(F_{da})_{y'} = \rho v^2 \sin\phi \cos\phi \sin^3\theta \, d\theta \, z'(h-z')\,dz'$$

$$y'(F_{da})_{z'} = r\sin\theta\,(F_{da})_{z'} = \rho v^2 \sin^2\phi \tan\phi \sin^3\theta \, d\theta \,(h-z')^2\,dz'$$

integrating with respect to θ:

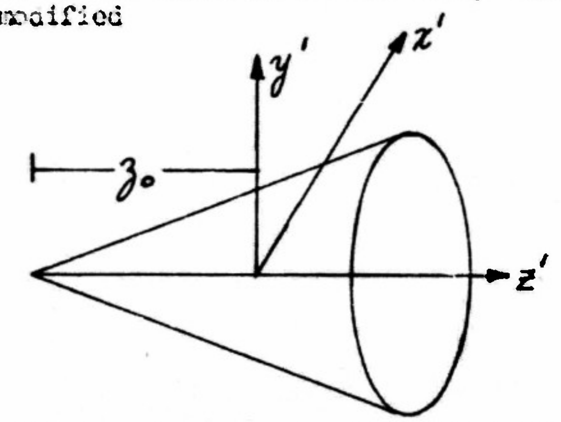
$$M_{x'} = \frac{4}{3}\rho v^2\left[-\sin^2\phi\tan\phi\int_0^h (h-z')^2\,dz' + \sin\phi\cos\phi\int_0^h z'(h-z')\,dz'\right]$$

$$= \frac{4}{3}\rho v^2\left[-\sin^2\phi\tan\phi\,\frac{h^3}{3} + \sin\phi\cos\phi\,\frac{h^3}{6}\right]$$

$$= \frac{2}{9}\rho v^2 h^3 \sin\phi\cos\phi\left[1 - 2\tan^2\phi\right]$$

The convention for sign that shall be adopted here is that a torque producing a counterclockwise motion is positive.

If the origin is in the interior of the cone, then the computation again has to be slightly modified



GEOMETRY FOR DETERMINING MOMENT WITH GENERAL POSITION OF CENTER OF ROTATION

Fig. C-5

Note that the contribution to the moment of $(z'-z_0)\,(F_{da})_{y'}$ changes sign when z' becomes greater than z_0.

The remainder of the computation is then fairly straightforward

DISTRIBUTION LIST

ASTIA (2)

WCAPP (2)

WCLE
Attn: Major J. H. Egy

WCLS
Attn: Mr. K. J. Wein

WCOWG
Attn: Mr. J. Yasecko

WCRRM

WCRT
Attn: Lt. Col. M. E. Sorte

Arnold Engineering Development Center
Tullahoma, Tennessee

Bell Aircraft Corporation
Niagara Falls, New York

Chance Vought Airplane Corp.
Attn: Librarian
Dallas, Texas
THRU: Bureau of Aeronautics
Navy Department
Washington 25, D. C.

Consolidated-Vultee Aircraft Corp.
San Diego Division
Attn: Mr. M. Rosenbaum
San Diego, California

Consolidated-Vultee Aircraft Corp.
Attn: Mr. E. S. Brown, Adm. Engr.
Fort Worth Division
Fort Worth 1, Texas

Douglas Aircraft Corporation, Inc.
Attn: Librarian
Santa Monica, California

Fairchild Engine and Aircraft Corp.
Pilotless Plane Division
Farmingdale, Long Island, N. Y.

Glenn L. Martin Company
Baltimore, Maryland

Hughes Aircraft Corporation
Division of Hughes Tool Company
Attn: Librarian
Florence Avenue and Teale Street
Culver City, California

Kirkland Air Force Base
Attn: Major R. E. Collier
Albuquerque, New Mexico

Lockheed Aircraft Corporation
Factory A, P.O. Box 551
Attn: Librarian
Burbank, California

McDonnell Aircraft Corporation
Attn: Librarian
St. Louis 21, Missouri

North American Aviation, Inc.
Los Angeles International Airport
Los Angeles 45, California

Northrop Aircraft, Inc.
Attn: Librarian
Hawthorne, California

Office, Chief of Ordnance
Department of the Army
Attn: ORDIR
Washington 25, D. C.

Republic Aviation Corporation
Farmingdale, Long Island, New York

Research and Development Division
Assistant Chief of Staff, G-4
Department of the Army
Attn: Development Branch
Washington 25, D. C.

Cy #1

53-WC-18724-1